Chapter 5 Continuous Functions

Let $f: A \to \mathbb{R}$ and $x_0 \in A$ (unlike in Ch. 4)
We say that f is continuous (cts.) at x_0 if
$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$(*) \quad |f(x) - f(x_0)| < \varepsilon \quad \forall x \in V_\delta(x_0) \cap A.$$

Two special cases:

(a) $x_0 \in A \setminus A^c$ (x_0 is an isolate pt. of A).
Then f is always cts at x_0. (because $\exists \delta_0 > 0$ s.t. $V_{\delta_0}(x_0) \cap A =$ singletone $\{x_0\}$).

(b) $x_0 \in A \cap A^c$. Then
f is cts at x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$

Th 1. Let $x_0 \in A$. For $f: A \to \mathbb{R}$, $\Leftrightarrow$:
(i) f is cts at x_0
(ii) $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

$$(**) \quad f(A \cap V_\delta(x_0)) \subseteq V_\varepsilon(f(x_0)).$$

Th 2 (Sequential Criterion for Continuity).

Let $x_0 \in A$. For $f : A \to \mathbb{R}$ TFAE:

(i) f is cts at x_0

(ii) $\lim_n f(x_n) = f(x_0)$ whenever $(x_n) \subseteq A$ with $\lim_n x_n = x_0$

Pf (i) $\Rightarrow$ (ii). Let $(x_n) \subseteq A$ with $\lim_n x_n = x_0$. Let $\varepsilon > 0$. Take $\delta > 0$ s.t. (*) holds. For this $\delta > 0$, $\exists N \in \mathbb{N}$ s.t. $|x_n - x_0| < \delta \ \forall n \geq N$, and it follows from (*) that

$$|f(x_n) - f(x_0)| < \varepsilon \quad \forall n \geq N,$$ showing (ii).

(ii) $\Rightarrow$ (i). Suppose (i) false. Then $\exists \varepsilon > 0$ s.t. each $\delta > 0$ fails (*) and so each $\frac{1}{n}$ $(n \in \mathbb{N})$ fails (*) with $\delta = 1/n$: $\exists x_n \in A$ with $|x_n - x_0| < 1/n$ but $|f(x_n) - f(x_0)| \geq \varepsilon$.

Then $\lim_n x_n = x_0$ but $f(x_n) \not\to f(x_0)$.

Examples

(a) Constant functions are cts: $f(x) = b \ \forall x \in \mathbb{R}$ say. Let $x_0 \in A := \mathbb{R}$. Then f is cts at x_0.

(b) $f(x) = x \ \forall x \in A := \mathbb{R}$. Then f is cts at x_0 for any $x_0 \in \mathbb{R}$.

(c) $f(x) = x^2 \ \forall x \in \mathbb{R}$. Then, $\forall x_0 \in \mathbb{R}$, f is cts at x_0.

Pf. Let $\varepsilon > 0$. Take $\delta := \min\left\{1, \frac{\varepsilon}{2|x_0|+1}\right\}$.

$x \in V_\delta(x_0)$. Need to show that

$$|f(x) - f(x_0)| < \varepsilon$$

To do this, note that

$$|f(x)-f(x_0)| = |x-x_0|\cdot|x+x_0| \leq |x-x_0|\cdot\left(|x-x_0|+2|x_0|\right)$$

$$\leq (1+2|x_0|)\,|x-x_0| < \varepsilon \quad \text{as was wished to show.}$$

Note. Our choice of δ depends on ε as well as x_0. (eventhough x_0 was arbitrary).

(d) $h(x) = 1/x \quad \forall\, x \in A := (0,+\infty)$. Let $x_0 \in A$, and $\varepsilon > 0$.

Take

$$\delta := \min\left\{\frac{|x_0|}{2},\ \frac{|x_0|^2}{2}\varepsilon\right\}.$$

Let $|x-x_0|<\delta$. Then $|x| \geq |x_0|/2$ and so

$$\left|\frac{1}{x}-\frac{1}{x_0}\right| \leq \frac{|x-x_0|}{\frac{|x_0|}{2}\cdot|x_0|} < \frac{\delta}{|x_0|^2/2} \leq \varepsilon$$

Note. Examples (c) & (d) are continuous but not uniformly continuous (to be studied in detail in §4).

(e) No matter how we extend the domain A to include $x=0$ by assigning the value at $x=0$ for the function h in (d), h will not be cts at the origin 0. Indeed, suppose not, that is, h cts at the new $x_0 = 0$ with $h(x_0) \in \mathbb{R}$. Let $\varepsilon := 1/2$. Then should h be cts at x_0, $\exists\, \delta > 0$ s.t.

$$|h(x)-h(x_0)| < \tfrac{1}{2} \quad \forall\, x \in V_\delta(x_0).$$

Pick $N \in \mathbb{N}$ s.t. $\frac{1}{N} < \delta$. Then

$$|h(\tfrac{1}{N}) - h(x_0)| < 1/2 \ \&\ |h(\tfrac{1}{2N}) - h(x_0)| < 1/2.$$

i.e.

$$|N - h(x_0)| < 1/2 \quad \& \quad |2N - h(x_0)| < 1/2$$

and so

$$N = |2N - N| < 1/2 + 1/2 = 1,$$

which is not possible for a natural no. N.

Similarly one can show that

(f). $f(x) = \begin{cases} 1 & \forall x > 0 \\ 0 & x = 0 \\ -1 & \forall x < 0 \end{cases}$

Then f is not cts at 0.

(g). $f(x) = \begin{cases} 1 & \forall x \in \mathbb{Q} \\ 0 & \forall x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. (Dirichlet Function)

Then f is <u>not cts</u> at any $x_0 \in \mathbb{R}$.

(same pf as in (e)).

(h). Let $A = (0, \infty)$ &

(Thomae function) $f(x) = \begin{cases} \frac{1}{n} & \text{if } x \triangleq \frac{m}{n} \in \mathbb{Q} \\ 0 & \text{if } 0 < x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ ("non-reducible" normal/canonical representation of x: $m, n \in \mathbb{N}$, without common factor other than 1)

Then f is cts at every irrational > 0
is not cts at every rational > 0.

Pf. Only prove the 1st assertion. Let $0 < x_0 \in \mathbb{R} \setminus \mathbb{Q}$ (so $f(x_0) = 0$)

Let

$$B := V_{\frac{1}{2}}(x_0) \cap \mathbb{Q} \cap (0, \infty)$$

$$= \{x \in V_{\frac{1}{2}}(x_0) : 0 < x = \frac{m}{n} \text{ with } m \in \mathbb{N}, n \in \mathbb{N}\}$$

$$= \bigcup_{n \in \mathbb{N}} B_n,$$

where $B_n := \{x \in V_{\frac{1}{2}}(x_0) : 0 < x = \frac{m}{n}$ with $m \in \mathbb{N}\}$

Note that each B_n is finite (as the set of all m appeared in the above definition of B_n is a bounded collection of natural nos.). Indeed you can alternative check that

(number of elements of B_n) $\#(B_n) \leq n$ (Exercise!) (#)

Let $\varepsilon > 0$. Take $N \in \mathbb{N}$ such that $\frac{1}{N} < \varepsilon$.

As $\bigcup_{n=1}^{N} B_n$ is a finite set not containing the irrational x_0 and so $\exists$ $\delta \in (0, \frac{1}{2})$ (so $V_\delta(x_0) \subseteq V_{\frac{1}{2}}(x_0)$) s.t. $V_\delta(x_0)$ is disjoint from $\bigcup_{n=1}^{N} B_n$ so any positive rational x in $V_\delta(x_0)$ can only be represented in the canonical form as

$$x \triangleq \frac{m}{n} \text{ with some } n > N$$

and hence $f(x) = \frac{1}{n} < \frac{1}{N} < \varepsilon$. This implies that

$$|f(x) - f(x_0)| = |f(x) - 0| < \varepsilon \quad \forall x \in V_\delta(x_0) \cap (0, \infty)$$

because also $f(x) - f(x_0) = 0 - 0 = 0$ $\forall$ irrational $x > 0$.